QFT in AdS instead of LSZ

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Outline

1. Introduction & Motivation
2. Setup, prescription & perturbative examples
3. Non-perturbative formulation:
    1. Conformal dispersion relations & analyticity
    2. Hyperbolic partial wave & unitarity

Introduction & Motivation

Analyticity of the S-matrix - Why?

- Dispersion relations:

$$\mathcal{T}(s,u) = \int_4^\infty ds' \frac{\text{Disc}_s[\mathcal{T}(s',u)]}{s'-s} + (s \leftrightarrow t)$$

- Bounds on the scattering amplitude. E.g. the Froissart-Martin bound.

$$\sigma_{\text{tot}} \leq C \log^2 s$$

[Froissart '61; Martin '63]

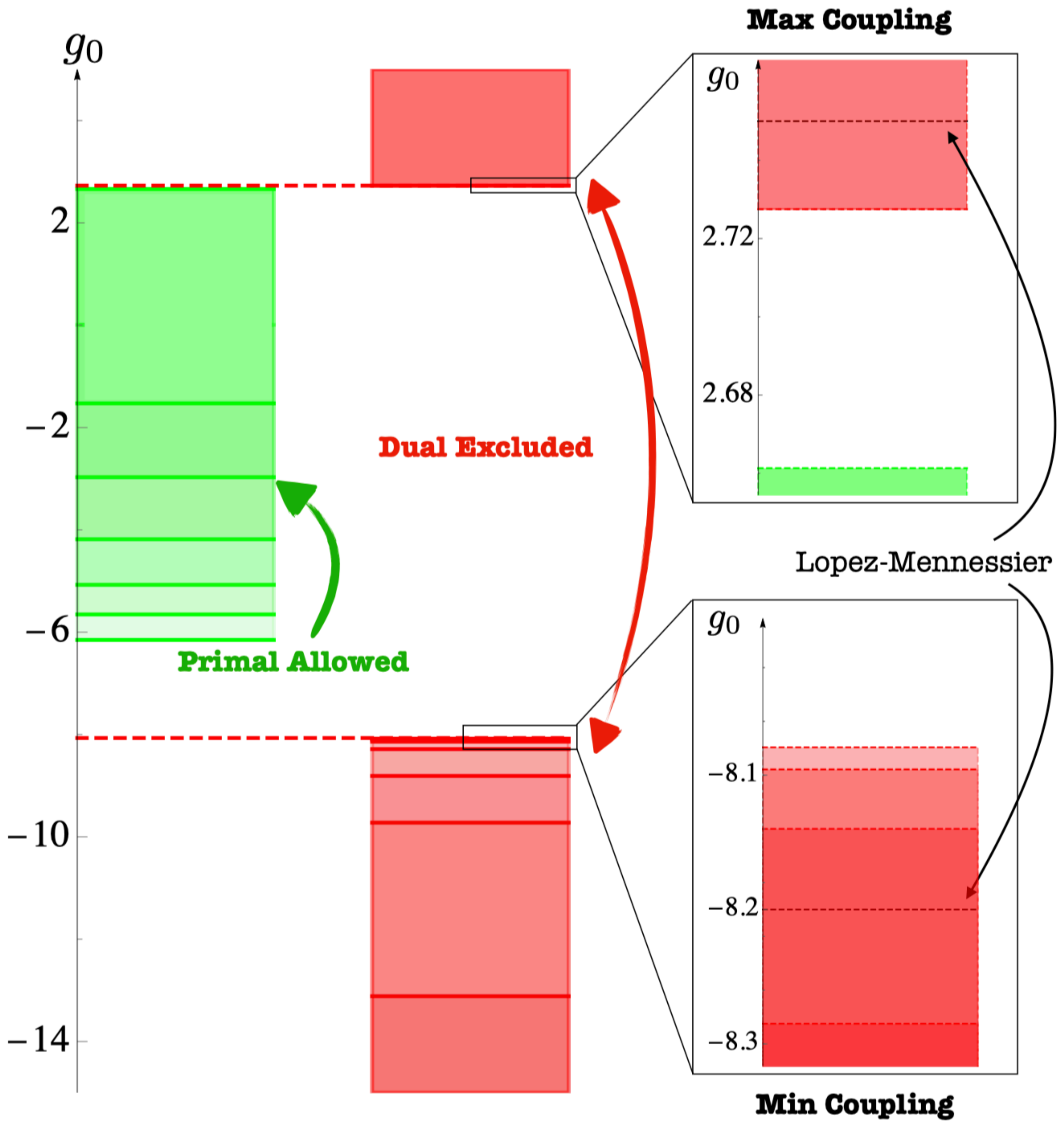
Analyticity of the S-matrix - Why?

- Numerical S-matrix bootstrap: carve out the space of S-matrices.

[Paulos, Penedones, Toledo, van Rees, Vieira '16; … Review: Kruczenski, Penedones, van Rees '22]

quartic coupling (4d) :

$$g_0 \propto M\left(\frac{4m^2}{3}, \frac{4m^2}{3}, \frac{4m^2}{3}\right)$$

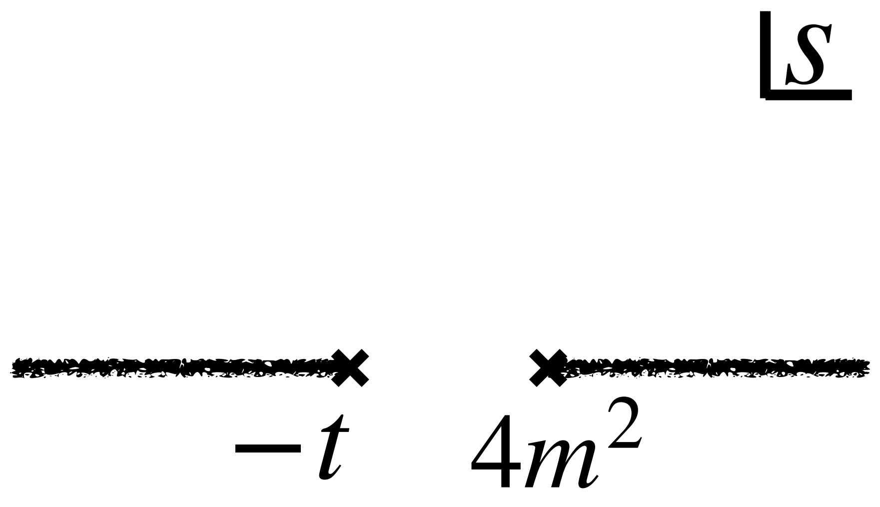


[Guerrieri, Sever '21]

Analyticity of the S-matrix - How?

- Axiomatic QFT.

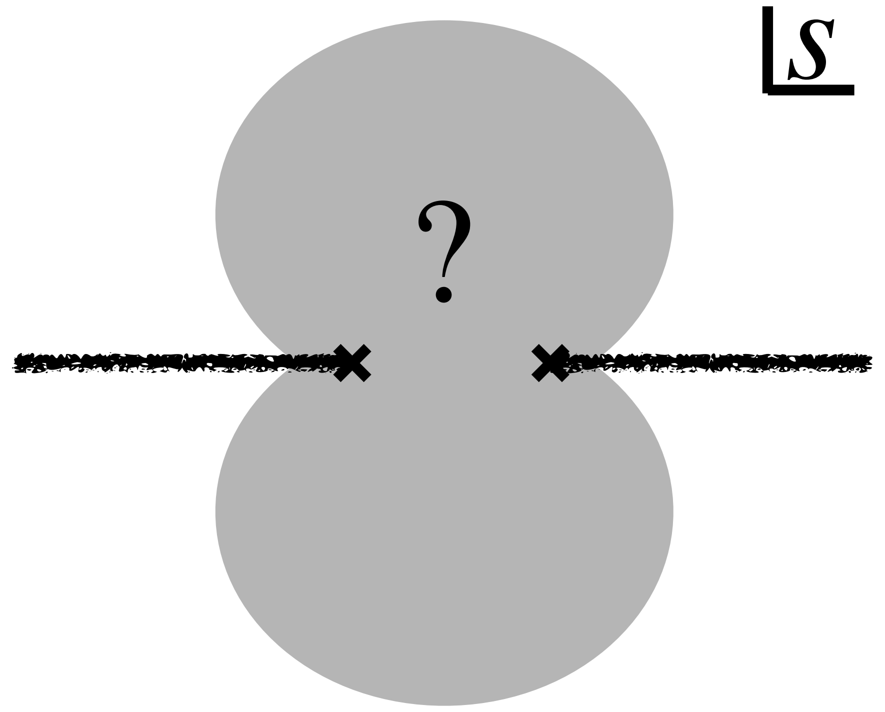
elastic 2-to-2 (lightest)



$$-28m^2 \leq t \leq 4m^2$$

[Martin ’65]

General 2-to-2



[Bros, Epstein, Glaser ’65]

- Perturbative analysis, *e.g.* Landau equations.
- The flat-space limit of QFT in AdS (focus of this talk).

[Paulos, Penedones, Toledo, van Rees, Vieira ’16; Dubovsky, Gorbenko, Mirbabayi ’17; Hijano ’19; Komatsu, Paulos, van Rees, XZ ’20; Li ’21; Gadde, Sharma ’22; van Rees, XZ ’22]

The flat-space limit of QFT in AdS

- Replace LSZ axioms (see e.g. [Bologubov, Logunov, Todorov, General Principles of Quantum Field Theory, 1990])

$$\langle \underline{\tilde{k}}_1 \ldots \underline{\tilde{k}}_a | S | \underline{k}_1 \ldots \underline{k}_b \rangle = \left[i \int d\tilde{x}_1 e^{-i\tilde{k}_1 \tilde{x}_1} (\Box + m^2) \ldots \right] \times \langle T\{\phi(\tilde{x}_1) \ldots \phi(\tilde{x}_a) \phi(x_1) \ldots \phi(x_b)\} \rangle$$

by CFT axioms + QFT in AdS

$$\text{S-matrix} \coloneqq \lim_{R \to \infty} \{\text{conformal correlation functions}\}_R$$

- Advantage:
  - Borrow the power of conformal symmetry, convergent OPE, state-operator correspondence...
  - Start with a sequence of well-understood analytic functions, rather than distributions.

Questions to answer

- The fate of the conformal correlators in the $R \to \infty$ limit?

- How to extract the S-matrix?

The Flat-space Limit Prescription

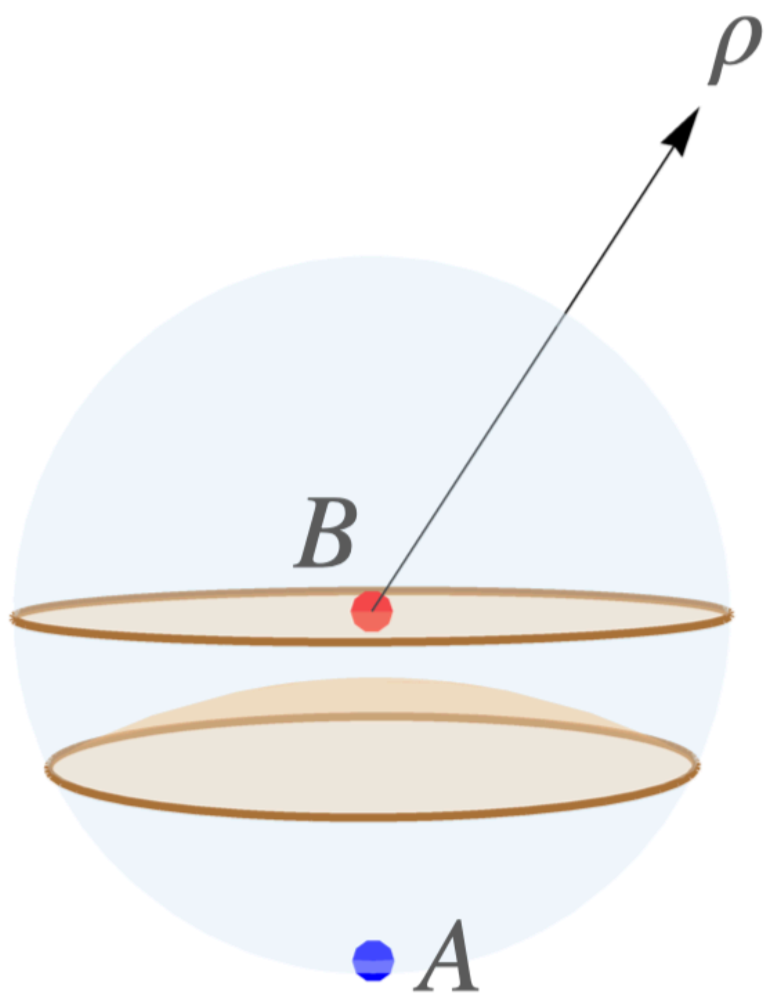
Preliminary: Euclidean AdS space

- Spherical coordinates: $ds^2 = d\rho^2 + R^2 \sinh^2\left(\frac{\rho}{R}\right) d\Omega_d^2$, $\rho \in [0, \infty)$ (R is AdS curvature radius)
  - $ds^2 \xrightarrow{R\to\infty} d\rho^2 + \rho^2 d\Omega_d^2$
- Global coordinates: $ds^2 = R^2 \frac{d\tau^2 + d\lambda^2 + \sin^2\lambda \, d\Omega_{d-1}^2}{\cos^2\lambda}$, $\lambda \in [0, \frac{\pi}{2}]$
  - $\tau = t/R$, $\lambda = r/R$, $R \to \infty$ again gives the flat-space metric.

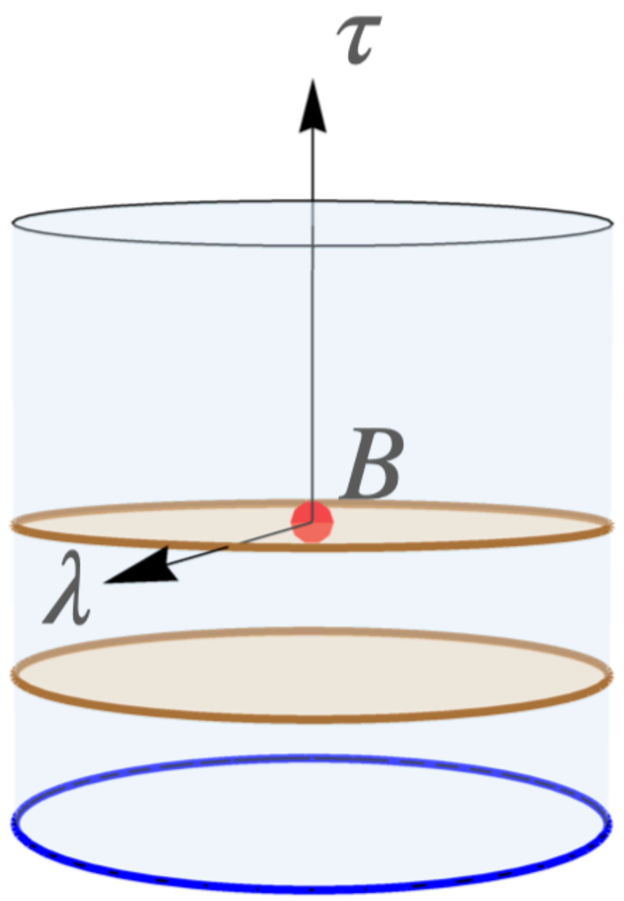
$X^A = (\rho, n_X^i)$, $|n_X| = 1$

$P^i = n_P^i$, $|n_P| = 1$

$i = 1, \ldots, d+1$



Spherical coordinates



Global coordinates

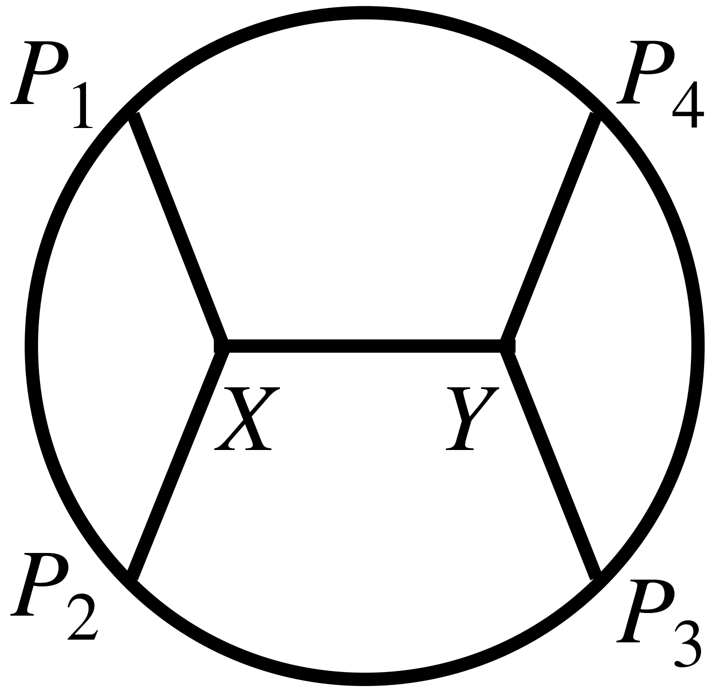
$X^A = (\tau, \lambda, n_X^i)$, $|n_X| = 1$

$P^i = (\tau, n_P^i)$, $|n_P| = 1$

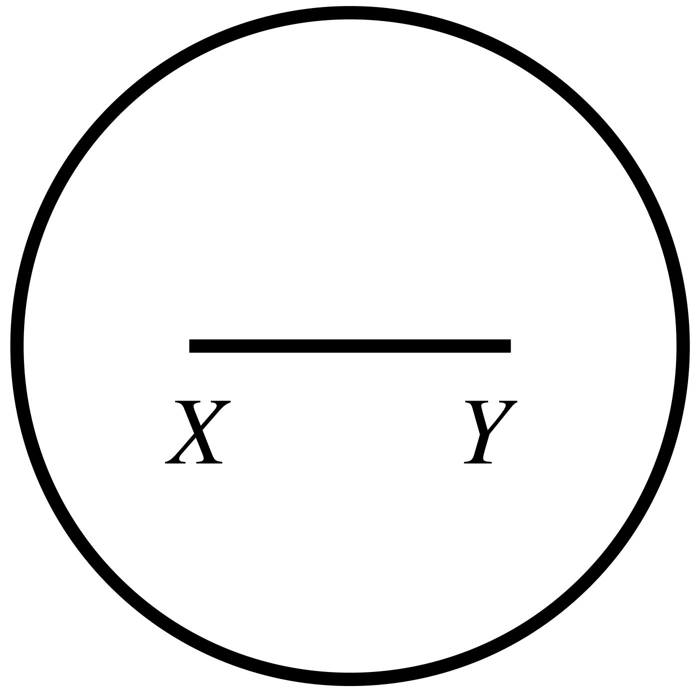
$i = 1, \ldots, d$

Preliminary: Witten diagram

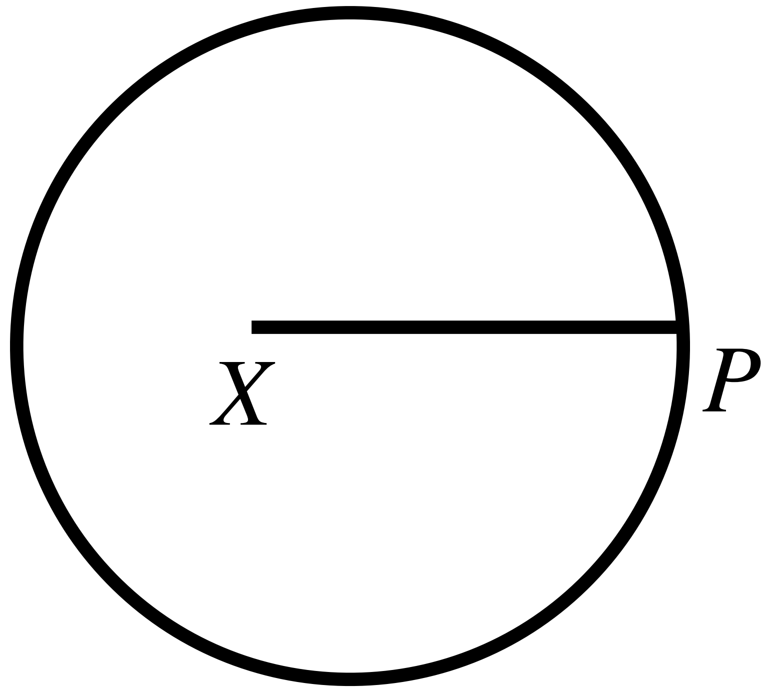
- Feynman diagram in AdS with external points pushed to conformal boundary at infinity.
- Two basic ingredients:
  - Bulk-bulk propagator: $(-\Box_X + m^2)G_{BB}(X,Y) = \frac{1}{\sqrt{g}}\delta^{(d+1)}(X-Y)$
  - Bulk-boundary propagator: $G_{B\partial}(X,P) = \lim_{\rho_Y\to\infty} e^{\Delta\rho_Y/R}G_{BB}(X,Y) = \frac{C_\Delta}{R^{(d-1)/2}(-2P\cdot X/R)^\Delta}$



Witten diagram



Bulk-bulk propagator



Bulk-boundary propagator

Main idea and setup

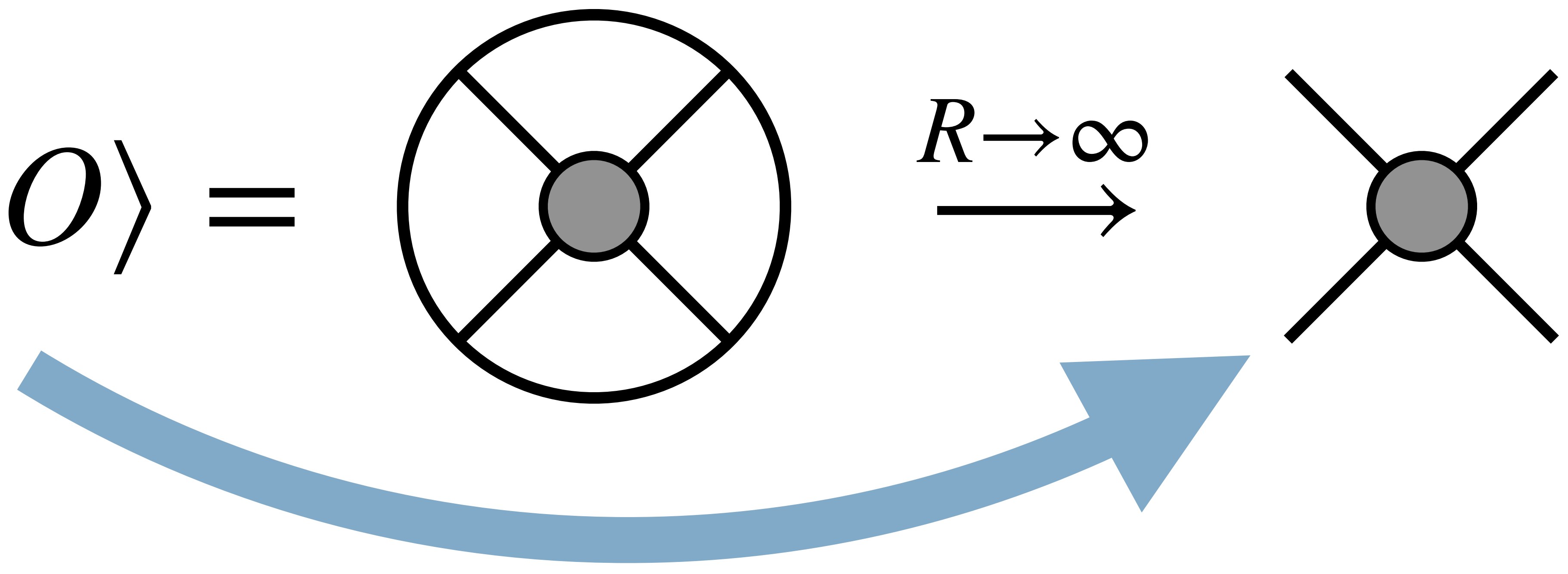
- Consider gapped QFT in *fixed* Euclidean AdS_{d+1} (~~gravity~~) with curvature radius R.

$$\{\phi,\, m\} \leftrightarrow \{O, \Delta\},\, \Delta(\Delta - d) = m^2R^2$$

- AdS isometry → boundary correlators are constrained by the conformal group $SO(d+1,1)$ and obey all the usual d-dimensional CFT axioms.

- Take the flat-space limit defined as: $\Delta \to \infty, R \to \infty, m = \dfrac{\Delta}{R}$ fixed

$$\langle OOOO \rangle =$$



analyticity, unitarity, boundedness...

The S-matrix conjecture

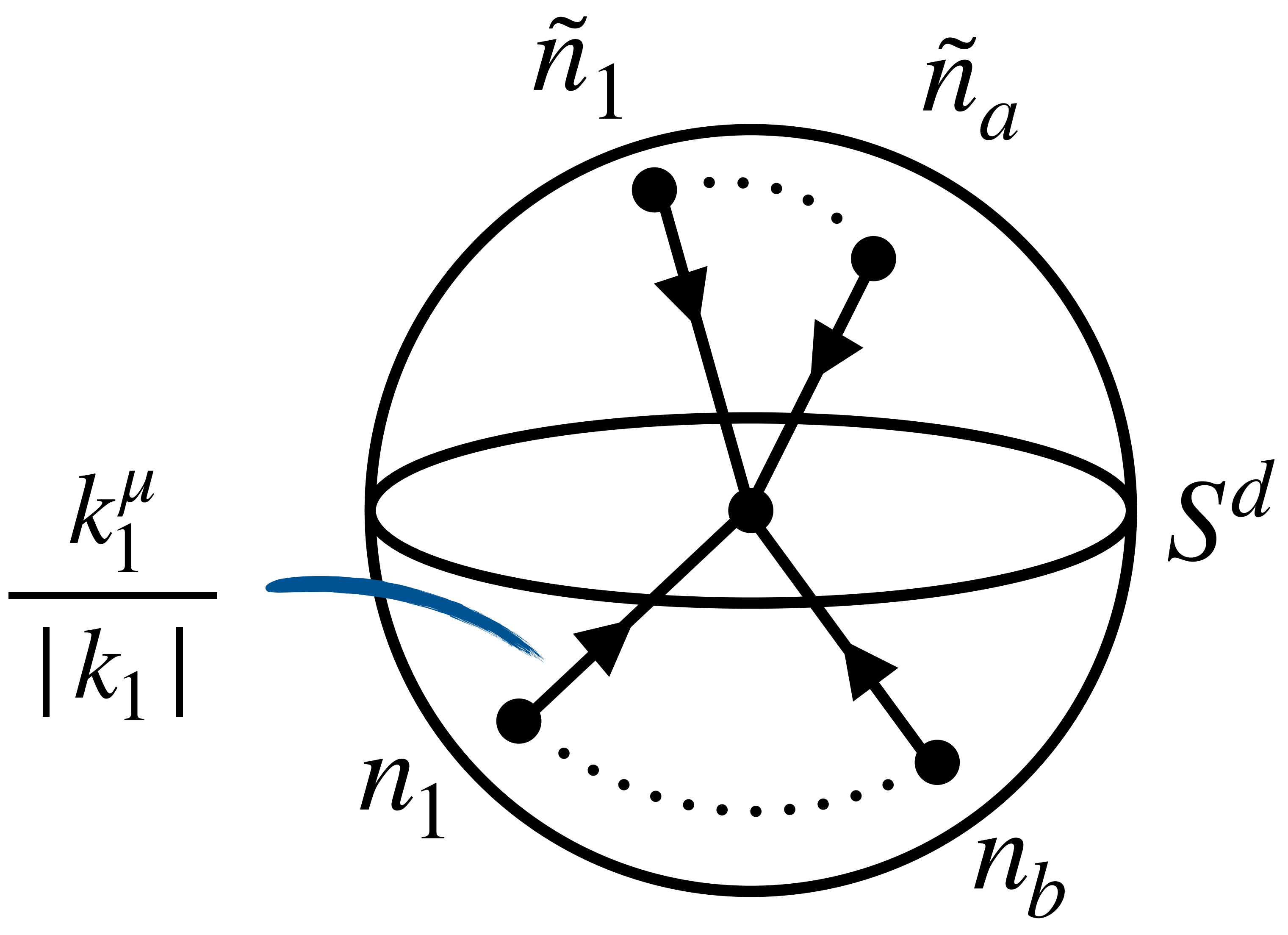
- The S-matrix conjecture:

$$\left\langle \underline{\tilde{k}}_1 \ldots \underline{\tilde{k}}_a | S | \underline{k}_1 \ldots \underline{k}_b \right\rangle \overset{?}{=} \lim_{R\to\infty} \langle \mathcal{O}(\tilde{n}_1) \ldots \mathcal{O}(\tilde{n}_a) \mathcal{O}(n_1) \ldots \mathcal{O}(n_b) \rangle \Big|_{\text{S-matrix}}$$

Flat-space S-matrix in $(d+1)$-dimension

Conformal correlator in d-dimension

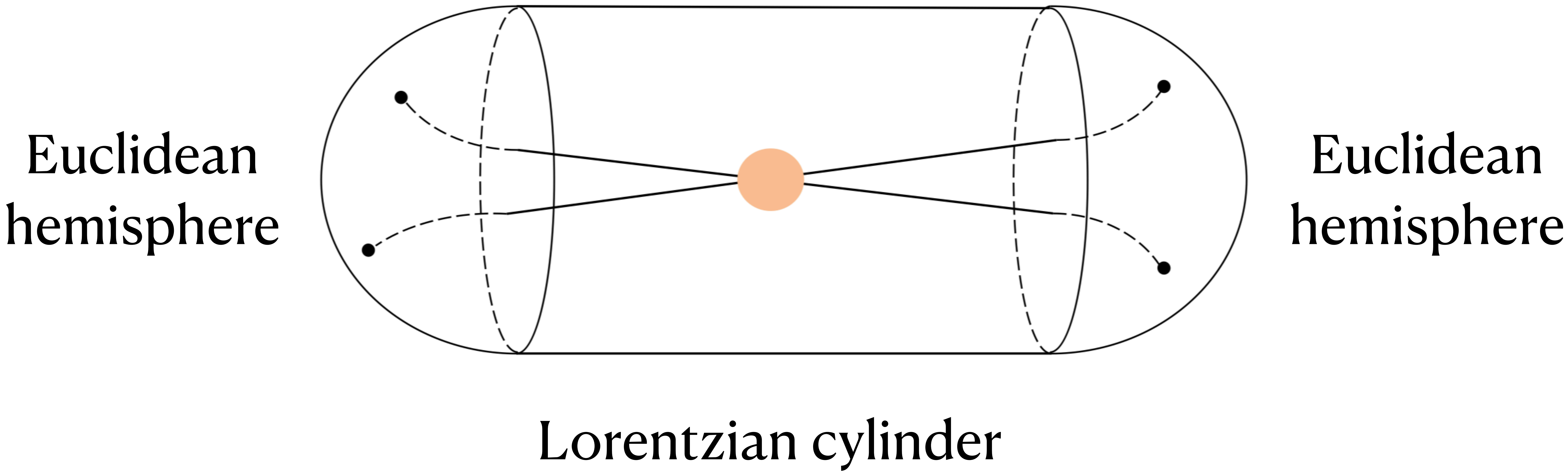
$(n^0, \underline{n}) = (-k^0, i\underline{k})/m$



Euc. AdS_{d+1} in spherical coordinates (before analytic continuation)

Visualising AdS in S-matrix configuration

[Hijano '19; Skenderis, van Rees '08]



This picture addresses the issue of building asymptotic states in Lorentzian AdS.

Example: 2-point function

- Take 2pt function and analytically continue to the S-matrix configuration with 1 incoming and 2 outgoing

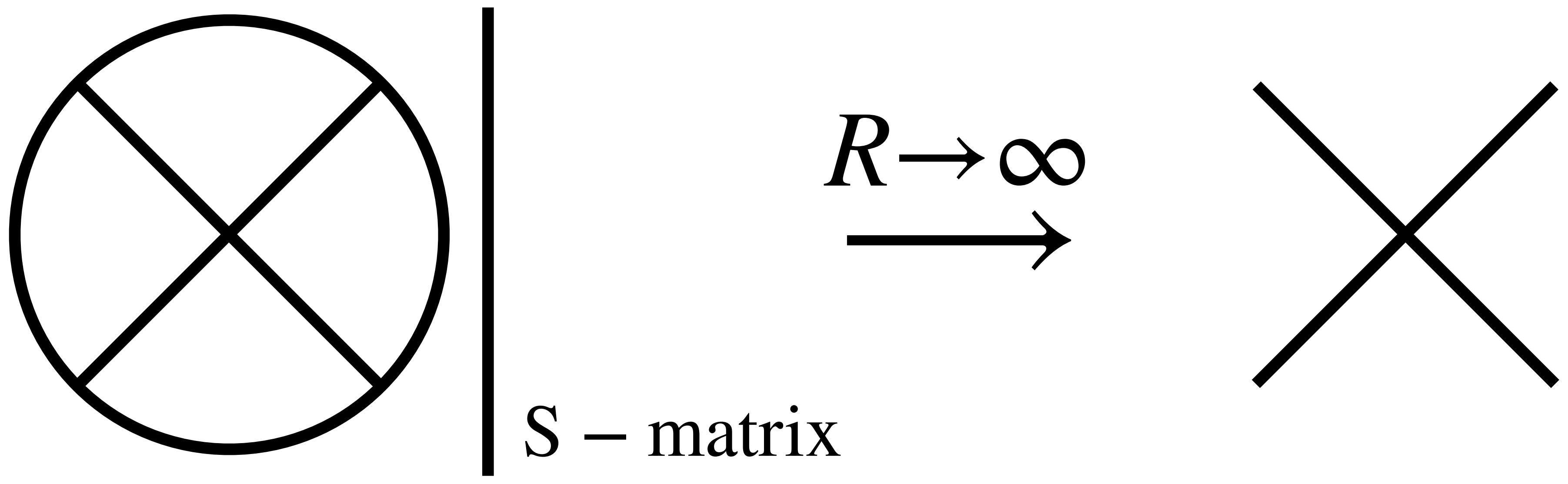
$$\langle \mathcal{O}_\Delta(n_1)\mathcal{O}_\Delta(n_2)\rangle \propto \frac{2^\Delta R^{d/2}}{(1-n_1\cdot n_2)^\Delta} = \frac{2^\Delta R^{d/2}}{(1-\cos\theta_{12})^\Delta}$$

$$\xrightarrow{\text{S-matrix config.}} \frac{2^\Delta R^{d/2}}{(1+\cosh\theta_{12})^\Delta}$$

$$\xrightarrow{\Delta\to\infty} 2E(2\pi)^d\delta^{(d)}\left(\underline{p}_1-\underline{p}_2\right)$$

Example: 4-point contact diagram

- Take 4pt contact Witten diagram and analytically continue to the S-matrix configuration with 1, 2 incoming and 3, 4 outgoing



$$= i(2\pi)^{d+1}\delta^{(d+1)}\left(k_1 + k_2 + k_3 + k_4\right)$$

The Amplitude conjecture

- The Amplitude conjecture:

$$\mathcal{T}\left(\tilde{k}_1 \ldots \tilde{k}_a ; k_1 \ldots k_b\right) \stackrel{?}{=} \lim_{R \to \infty} \left. \frac{\langle \mathcal{O}\left(\tilde{n}_1\right) \ldots \mathcal{O}\left(\tilde{n}_a\right) \mathcal{O}\left(n_1\right) \ldots \mathcal{O}\left(n_b\right) \rangle_{\text{conn.}}}{\mathcal{G}_c\left(\tilde{n}_1, \ldots, \tilde{n}_a, n_1, \ldots, n_b\right)} \right|_{\substack{\text{S-matrix,} \\ \text{mom. cons.}}}$$

Contact Witten diagram

$$\text{Recall: } S = \text{disconn.} + i(2\pi)^{d+1} \delta^{(d+1)}\left(\sum_i \tilde{k}_i + \sum_j k_j\right) \mathcal{T}\left(\tilde{k}_1 \ldots \tilde{k}_a ; k_1 \ldots k_b\right)$$

Advantage: valid even in unphysical regions (e.g. Mandelstam $s \in \mathbb{C}$ in 2-to-2 scattering).

Conformal Mandelstam invariants

- Boundary points ↔ bulk momenta ⇒ cross ratios ↔ Mandelstam invariants
- For 2-to-2 scattering of identical particles, define the *conformal* Mandelstam invariants s, t, u (essentially cross ratios):

$$s(r) = 4\left(\frac{1-r}{1+r}\right)^2, \quad t(r,\eta) = \frac{8r}{(1+r)^2}(1+\eta), \quad u = 4 - t - s$$

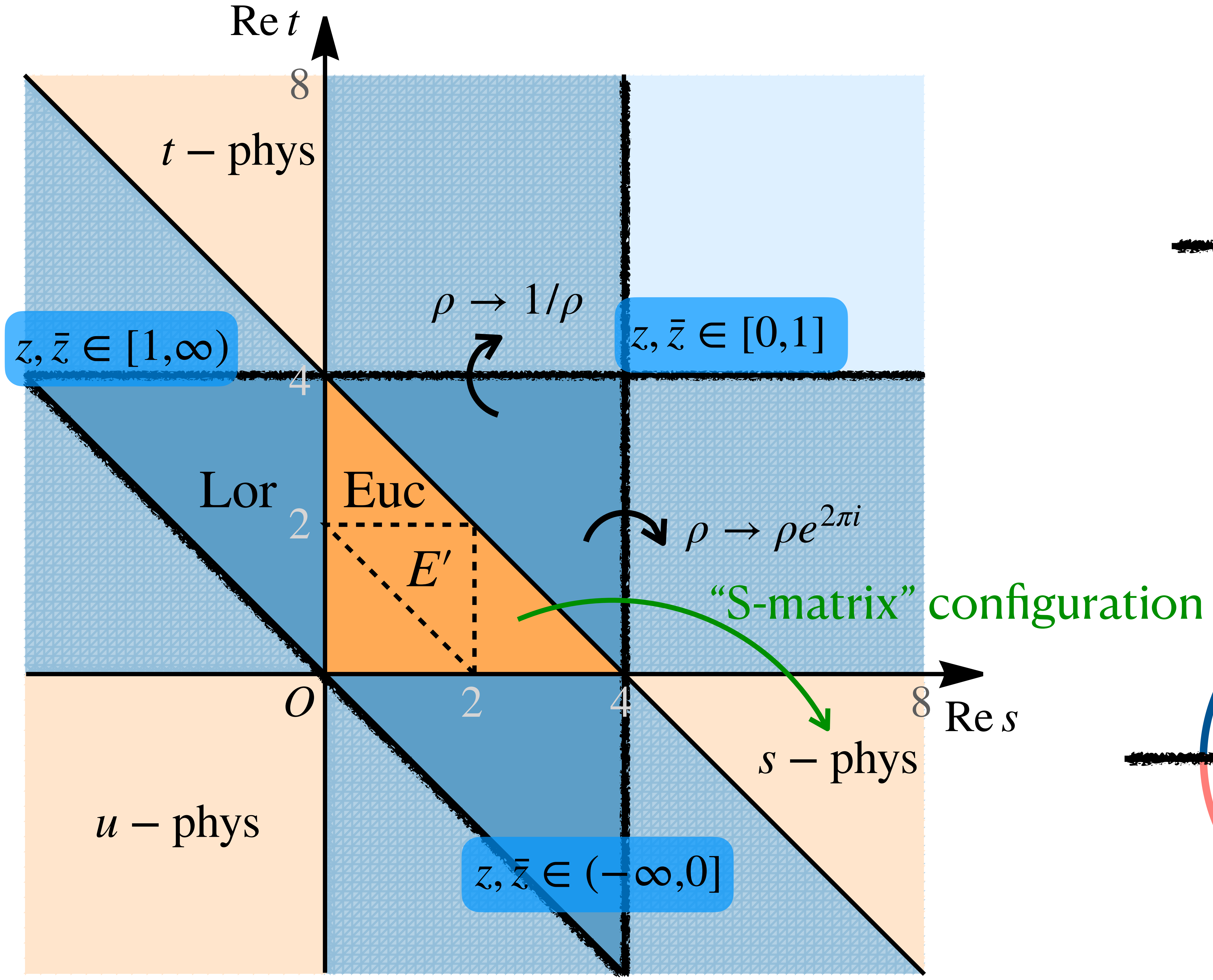
where $r = \sqrt{\rho\bar{\rho}}, \quad \eta = (\rho + \bar{\rho})/(2\sqrt{\rho\bar{\rho}})$

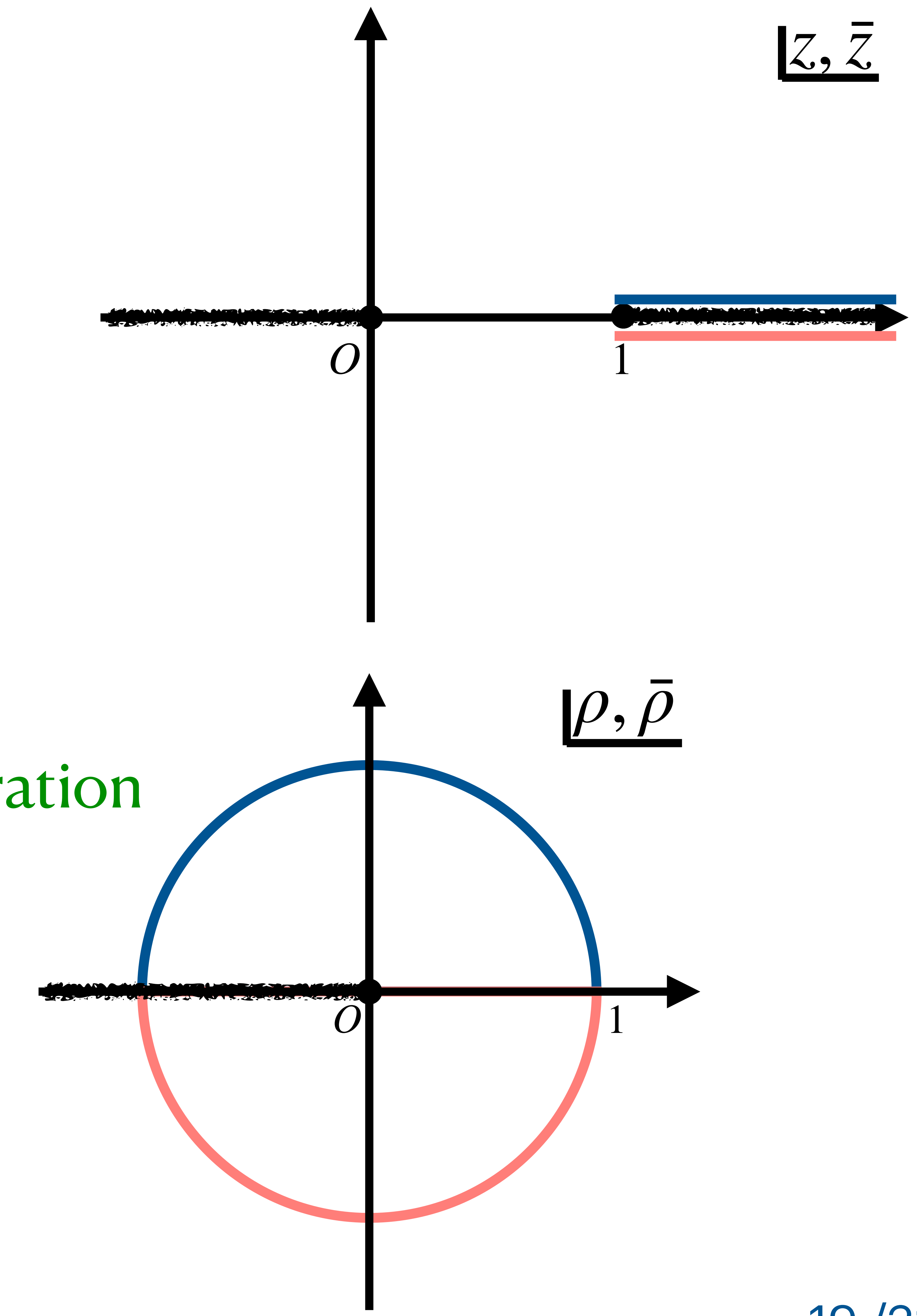
- Identified with the usual Mandelstam invariants in the flat-space limit ($m = 1$).

Conformal Mandelstam plane

Dark: 1st sheet
Shallow: 2nd sheets

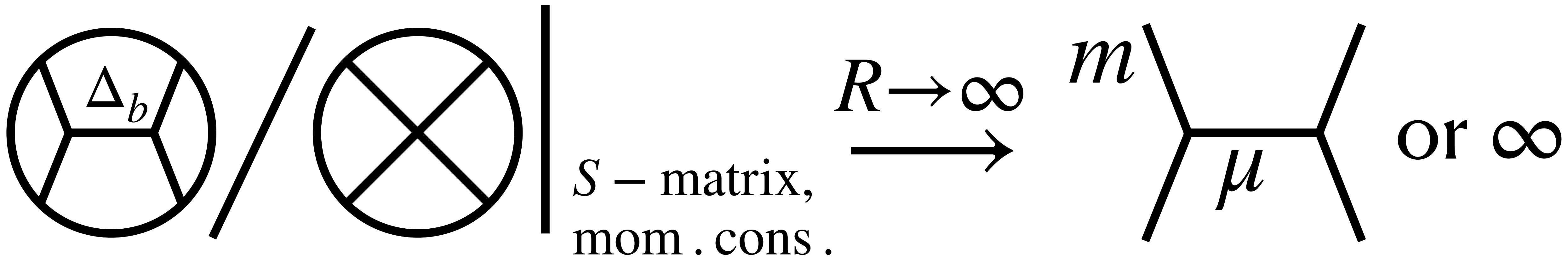
Orange: Euclidean
Blue: Lorentzian



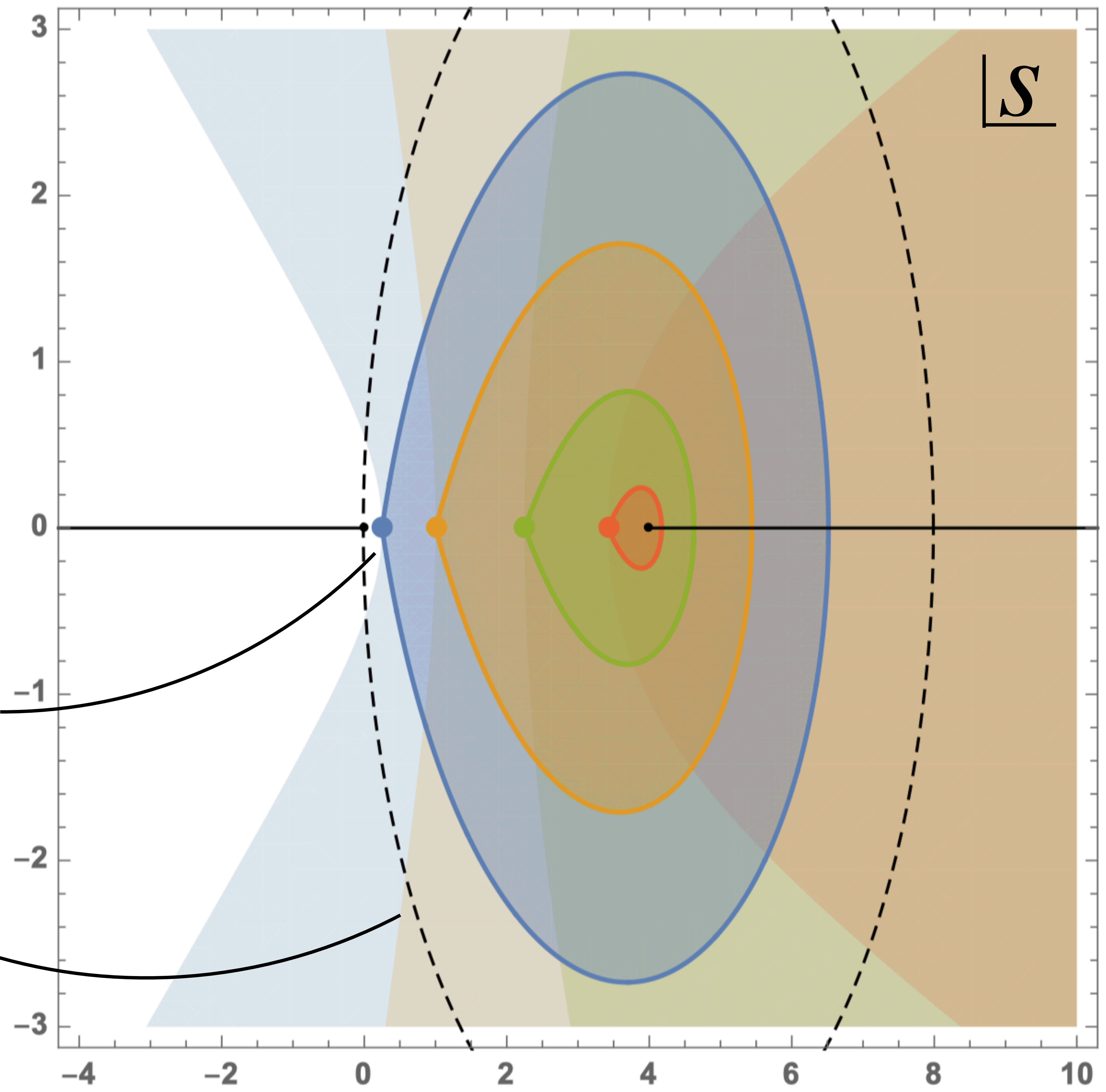


Example: scalar exchange diagram

Using the amplitude conjecture:



$$\frac{\mu}{m} = \frac{\Delta_b}{\Delta} = 0.5, 1, 1.5, 1.85$$



the flat-space pole at $s = \mu^2$

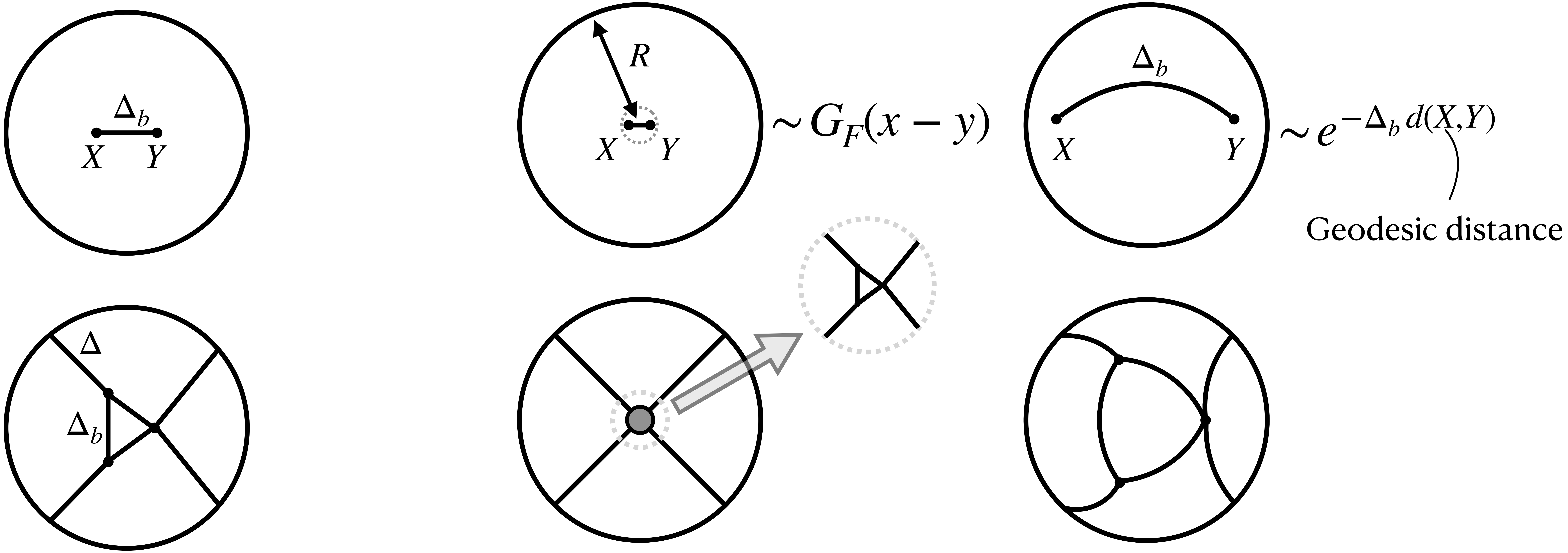
$|s - 4m^2| = 4m^2$

Divergence comes from the single trace block of the exchange Witten diagram.

The starting point of divergence is controlled by the OPE spectrum.

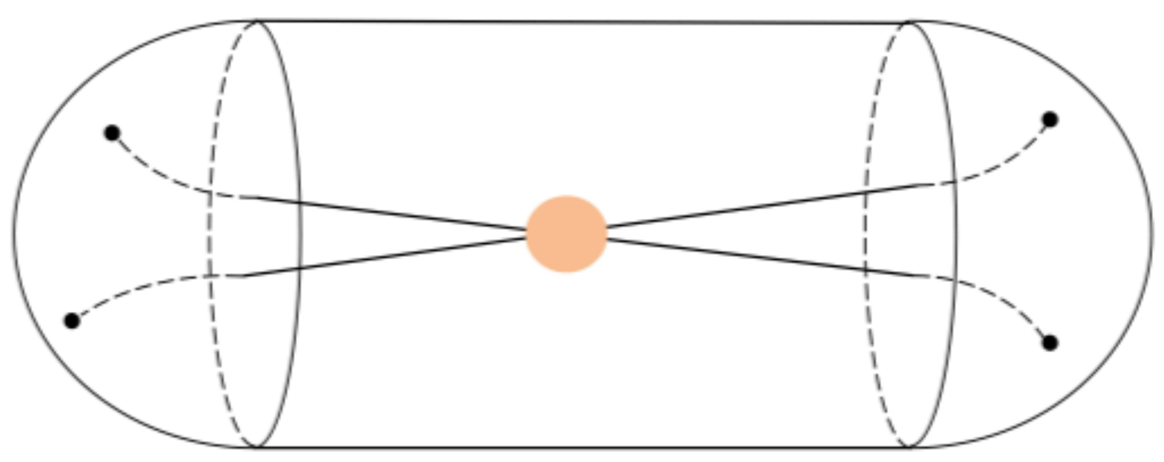
G_{BB} is the key to understand divergences

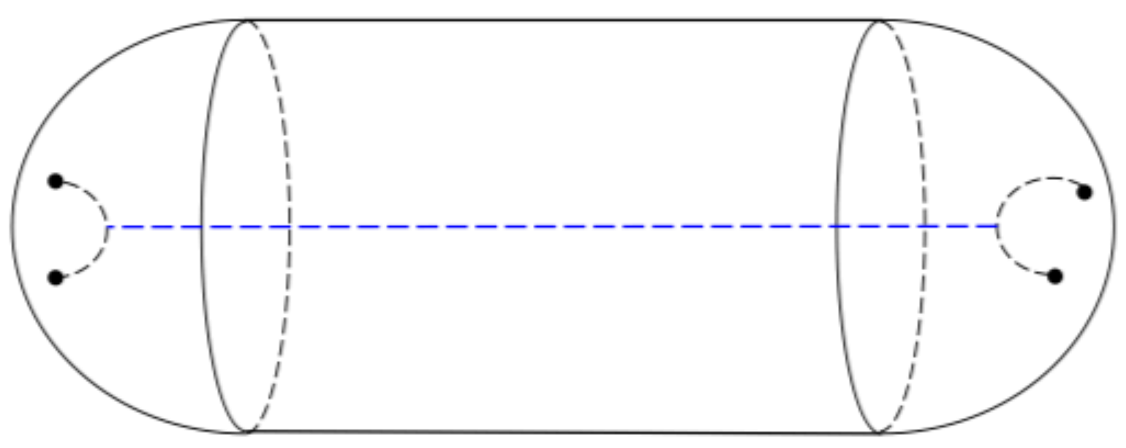
Witten diagram	$\xrightarrow{R\to\infty}$	Feynman diagram $(X-Y)^2 \ll R^2$	+	**AdS Landau diagram** (not always present) $(X-Y)^2 \sim R^2$



AdS Landau diagram

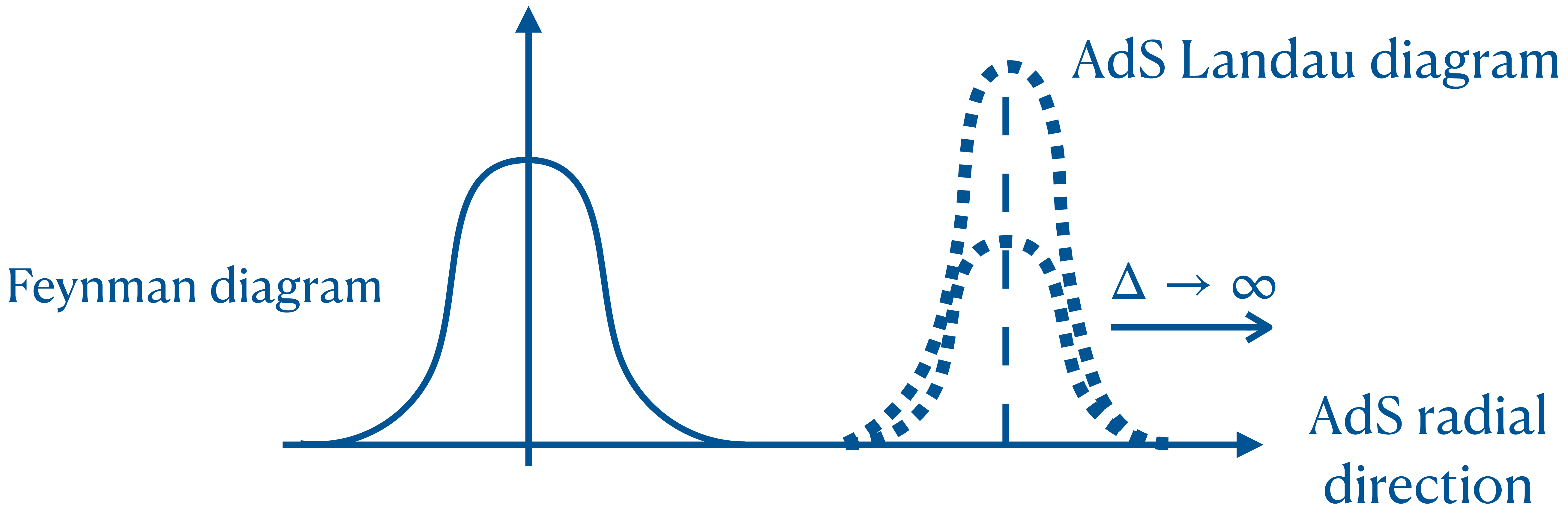
- *AdS Landau diagrams* are Witten diagrams with the bulk-bulk propagators $G_{BB}(X_i, X_j)$ replaced by $e^{-\Delta_{ij} d(X_i, X_j)}$, and bulk points located at their saddle points.
- Our conjectures work when Feynman diagrams dominate over AdS Landau diagrams (if present).





A simple way to remove divergences

- When calculating flat-space limit of Witten diagrams one *cannot* exchange the limit and the integral.
- “Lumps of mass” sliding to infinity:

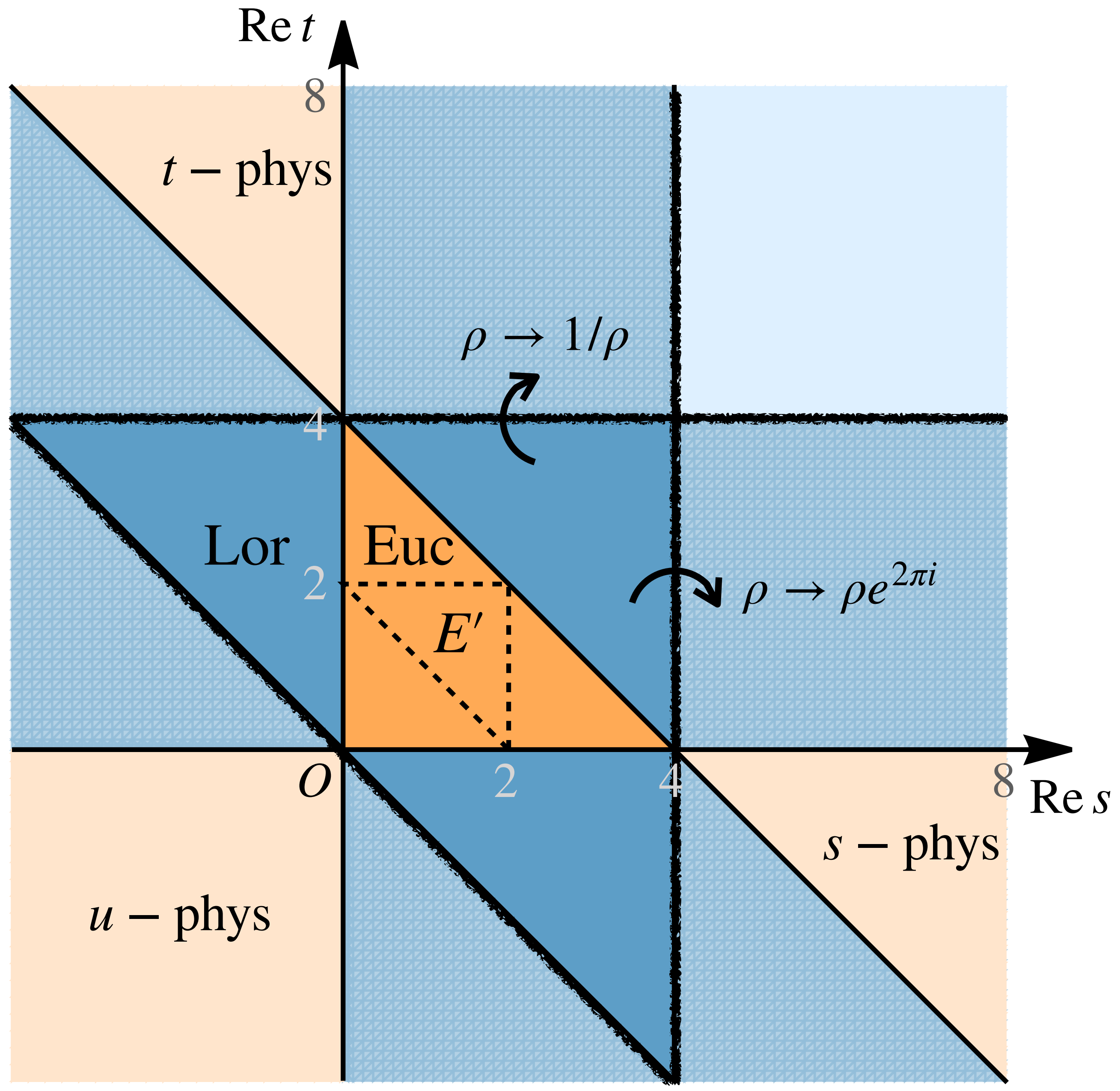


- By swapping the limit and the integral we can get the wanted answer. The non-trivial step is retaining nice properties of the original conformal correlator.

Towards Non-perturbative Formulation

Two main challenges

1. Remove the **disconnected correlator**.
   - Straightforward subtraction destroys positivity given by CFT unitarity.
2. Avoid **AdS Landau diagrams** in *all channels*.
   - Fixing one channel is easy, but more delicate with all channels.



Non-perturbative analyticity

- Object: $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = x_{12}^{-2\Delta_\phi}x_{34}^{-2\Delta_\phi}\mathcal{G}(s,t,u)$
- Assumptions:
  1. $\Delta > \sqrt{2}\Delta_\phi$ (to fix *u*-channel AdS Landau diagrams)
  2. The flat-space limit *exists* in

$$E' = \{(s,t,u)|s,t,u \le 2,\ s+t+u=4\}$$

- Main tool: conformal dispersion relation

$$\mathrm{dDisc}_s[1_{t,u}] = 0$$

$$(z\bar{z})^{-\Delta_\phi}\underbrace{[\mathcal{G}(z,\bar{z})-\mathcal{G}_{\text{gff}}(z,\bar{z})]}_{\mathcal{G}_{\text{conn}}(s,t,u)} = \iint dw d\bar{w}\, K_2(z,\bar{z};w,\bar{w})\,\mathrm{dDisc}_s[(w\bar{w})^{-\Delta_\phi}(\mathcal{G}(w,\bar{w})-1_s)]$$

$$+\,((z,\bar{z})\leftrightarrow(1-z,1-\bar{z}))$$

$$\mathcal{G}_{\text{gff}} := 1_s + 1_t + 1_u$$

Non-perturbative analyticity

- Advantage:
  - Retains positivity even for connected correlators.
- Result:
  - (Subtracted) Dispersion relation in s for fixed $u \in [0,2m^2]$

$$\frac{T^{\text{sub}}(s_1,u) - T^{\text{sub}}(s_2,u)}{(s_1-s_2)(s_1+s_2+u-4)} = \sum_\ell \int d\mu \frac{\rho_\ell(\mu)\left(2\mu^2+u-4\right) P_\ell^{(d)}\left(-1+\frac{2u}{4-\mu^2}\right)}{(s_1-\mu^2)(s_2-\mu^2)(4-u-s_1-\mu^2)(4-u-s_2-\mu^2)}$$

  - $\implies$ Non-perturbative S-matrix analyticity in s modulo branch cuts
  - Also: extended unitarity

Non-perturbative unitarity

- Recall partial wave coefficients in S-matrix (actual Mandelstam):

Normalisation

$$f_\ell(s) = \frac{\mathcal{N}_d}{2} \int_{-1}^{1} d\eta \left(1-\eta^2\right)^{\frac{d-3}{2}} P_\ell^{(d)}(\eta) \mathcal{T}(s, t(s,\eta))$$

cos(scatt. angle)

  - The unitarity condition reads:

$$\left|1 + is^{-1/2}(s-4m^2)^{d/2-1} f_\ell(s)\right| \leq 1$$

- Define "hyperbolic partial waves" $c_\ell(s)$ by replacing $\mathscr{T}$ with $\mathscr{G}/\mathscr{G}_c$ *

- Combine with CFT unitarity:

$$|\mathcal{G}_{\text{gff}} + \mathcal{G}_{\text{conn}}|_{\text{2nd sheet}} \leq |\mathcal{G}_{\text{gff}} + \mathcal{G}_{\text{conn}}|_{\text{1st sheet}}$$

0

Conclusion

Conclusion

- Prescriptions to extract S-matrix and scattering amplitude from conformal correlators.
- "S-matrix" analytic continuation configuration and scattering picture in Euc-Lor-Euc AdS.
- Conformal Mandelstam variables.
- AdS Landau diagrams and its subtraction.
- Non-perturbative analyticity and unitarity

Conclusion

	QFT in AdS_{d+1} $(R \to \infty)$	CFT_d
S-matrix/Correlator	$S = \delta^{(d)}(k_1 - k_3)\delta^{(d)}(k_2 - k_4) + (3 \leftrightarrow 4) + i(2\pi)^{(d+1)}\delta^{(d+1)}\left(\sum_i k_i\right)\mathcal{T}(s,t,u)$	$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$
Amplitude/ connected correlator	$\mathcal{T}(s,u)$	$\dfrac{\mathcal{G}_{\mathrm{conn}}(z,\bar{z})}{\mathcal{G}_c(z,\bar{z})}$
Subtracted dispersion relation	$\mathcal{T}(s,u) - g(u) = \frac{1}{\pi}\int_4^\infty ds' \, \frac{s^2}{s'^2(s'-s)} \mathrm{Disc}_s[\mathcal{T}(s',u)] + (s \leftrightarrow t)$	$\mathcal{G}_{\mathrm{conn}}(z,\bar{z}) = \int dw d\bar{w} K_2(z,\bar{z};w,\bar{w}) \mathrm{dDisc}_s\left[\mathcal{G}(w,\bar{w}) - 1_s\right] + ((z,\bar{z}) \leftrightarrow (1-z, 1-\bar{z}))$
Unitarity	$\left\lvert 1 + i\frac{(s-4m^2)^{\frac{d-2}{2}}}{\sqrt{s}} f_\ell(s)\right\rvert \leq 1$	$\lvert \tilde{c}_\ell(s)\rvert \geq \lvert c_\ell(s)\rvert$